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Statistical mechanics of elastica on a plane: origin of the MKdV hierarchy

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Abstract. In this paper, the statistical mechanics of a non-stretching elastica in two-dimensional space using the path integral method is investigated. In the calculation, the modified Korteweg–de Vries (MKdV) hierarchy naturally appeared in the equations including the temperature fluctuation. We have classified the moduli of the closed elastica in a heat bath and summed the Boltzmann weight with the thermal fluctuation over the moduli. Due to the bilinearity of the energy functional, its exact partition function has been obtained. By investigation of the system, it is conjectured that an expectation value at a critical point of this system obeys the Painlevé equation of the first kind and its related equations are extended by the Korteweg–de Vries hierarchy. Furthermore, we also comment on the relation between the MKdV hierarchy and the Becchi–Rouet–Stora transformation in this system.

1. Introduction

Elastica has sometimes appeared in the history of mathematical physics according to [1–4]. The problem of elastica, an ideal thin elastic rod, was proposed by James Bernoulli in 1691. By their investigations on the behaviour of an elastica, Bernoulli's family and related people, Euler, d'Alembert and so on, discovered many non-trivial mathematical and physical facts, for example, classical field theory, minimal principle, elliptic function, mode analysis, nonlinear differential equations etc [1–4]. In fact, James Bernoulli derived the elliptic integral related to the lemniscate function in 1694, before Fagnano found his lemniscate function [1–3], and found that the force of elastica is proportional to the inverse of its curvature radius [1]. His nephew, Daniel Bernoulli followed James' discoveries and discovered the energy functional of elastica and its minimal principle around 1738. Succeeding these discoveries, Euler derived the elliptic integral of general modulus as a shape of elastica using Daniel's minimal principle and numerically integrated it. Then he completely classified shapes of the static elastica by a numerical sketch [1], which are, nowadays, known as special cases of the loop soliton [5]. Even though in the computations Euler did not directly use the static sine–Gordon equation or the static modified Korteweg–de Vries (MKdV) equation, these computations essentially imply discovery of the integrable nonlinear differential equation, the investigation of its moduli and first application of algebro-geometrical functions to physics. It should be noted that they came from the studies on the elastica. Furthermore, it is well known that the elastica problem is the simplest prototype σ-model $S^1 \to \mathrm{SO}(2)$ or $\mathrm{SO}(3)$ [6, 7], which was found in the 18th century and investigated by Kirchhoff in the last century [4].

Thus the elastica problem seems to be a legacy from before the last century but its properties are not completely understood and its role in mathematical physics is still

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important. The difficulty in solving the elastica problem is that one must consider the constraint condition such as the isometry (non-stretch) condition and boundary condition. In fact, even though it is not an elastica problem, Goldstein and Petrich naturally rediscovered the MKdV hierarchy through (virtual) dynamics of a space curve with isometry condition [8, 9]. (Readers should note that the time-development of the physical elastic rod is not governed by the MKdV hierarchy in general [7, 9–14] even though in some papers it seems to be misunderstood.) They showed that the MKdV hierarchy can be geometrically realized. Following their works, new geometrical interpretations of soliton theory using the space curves were developed ([15] and references therein). For example, Doliwa and Santini gave a new construction of the inverse scattering method with non-vanishing spectral parameters using a space curve in an n-dimensional sphere [15]. (In [15], the history of the (new and old) studies on the relations between soliton theory and isometry space curves are also described in detail.) Furthermore, using the Goldstein–Petrich construction, it has been proved that the Hirota bilinear equation and the τ function of the MKdV hierarchy [16] can also be translated into the geometrical language of a non-stretching space curve [11]. The classical analogue of the vertex operator naturally appears as a complex tangent vector of the curve [11].

However, these approaches are just mathematical and geometrical ones but are not directly connected with the physical problem because in most of the works there is little physical reasoning why the MKdV hierarchy must appear in the physical system [8, 9, 15]. In other words, they were able to show that constituents of the soliton theory are connected with the space curve geometry but could not explain why they are gathered and become the soliton equation. In [11], we partially pointed out that taking into account that the soliton theory is a kinematic object and the minimal principle (stationary principle) is essential, one should not deal with a mathematical space curve but should investigate a curve with energy functional or an elastica. Thus the first purpose of this paper is to present a physical answer to the question by considering the statistical mechanics of elastica.

On the other hand, the study of an elastic chain model of a large polymer like deoxyribonucleic acid (DNA) is current in polymer science [17–24]. In the case of DNA, it usually occurs as a double helix with two complementary nucleotide chains winding around a common axis and the common axis of the looped DNA often folds into intricate structure, or a supercoiled form [17]. Due to its enormous size, conformation of the double helical polymer needs topological and geometrical studies [17–21]. Thus the double helical polymer and, similarly, other large polymers [23] are often modelled as thin elastic isotropic rods or an elastica and studied from the kinematic and topological point of view. Topological invariance such as the linking number classifies the shape of the large polymer in three-dimensional Euclidean space $\mathbb{R}^3$ [17]. Based on the Kirchhoff model of an elastica in $\mathbb{R}^3$ and the nonlinear Schrödinger equation, their possible kinematic conformations were considered [17–20]. Partial thermal effects on their shapes were argued in [18]. Furthermore, a molecular dynamics study combining with the elastic energy model was reported [21] and then a topology change related to the knot was found [21, 22].

Statistical mechanics of the elastic chain as a model of a large polymer like DNA was studied by Saitô *et al* using the path integral [23, 24]. They calculated some exact partition function of the energy functional of the elastic chain. Such a chain model is sometimes called a wormlike chain model [25]. Their approach is successive in polymer physics and influences recent works ([26] and references therein). However, they did not pay any attention to the isometry condition in calculation of the path integration even though they imposed an isometry condition after computation (a kind of quantization); they summed over all configuration space without an isometry condition. It should be noted that the constraint does not commute with quantization and statistical treatment [27]. (An example of such inconsistency is shown in [28].)

It is a natural assumption that flexible polymers which cannot stretch as a classical object (at zero temperature) cannot stretch even in a heat bath in the classical regime, i.e. neglecting the quantum effect. Hence, it is very important to calculate the partition function of the elastica with isometry conditions and sum the Boltzmann weight only over allowed conformations.

The second purpose of this paper is to investigate the behaviour of an elastica, as a model of a closed large polymer like DNA, in a heat bath of two-dimensional space, whose shape is determined by its elasticity and stretch is negligible. As will be shown later, this investigation is identified with the first purpose to clarify the physical meaning of the MKdV hierarchy.

We wish to note that in the case when the elastic torsion force of an elastica in three-dimensional space is negligible, the conformation of the elastica is expressed by the nonlinear Schrödinger equation [17–21]. Using the similarity between the MKdV equation and the nonlinear Schrödinger equation, the system we will deal with can be extended to the three-dimensional system [15, 28]. However, due to the dimensionality, handling the MKdV equation is easier than calculating the nonlinear Schrödinger equation. Thus in this paper, we investigate only the elastica in a plane as a first step, and thanks to two-dimensionality, give rigorous arguments for it. However, as long as a polymer is in the region of the phase diagram in which its elastic torsion force can be neglected, this assumption with respect to two-dimensionality is not so far from physics as we discuss in the final section.

Furthermore, it is known that DNA sometimes exhibits topology changes and has a topological isomer [22, 29]. In this paper, as we deal with an elastica in two-dimensional space, there is no knot invariance but there exists a writhing number as a topological invariance of a curve in two-dimensional space if crossings are allowed [28]; the writhing number corresponds to the linking number of a curve in three-dimensional space. Thus in this paper, even though the elastica in two-dimensional space will be investigated, we will allow the crossings if they can be realized when embedded in three-dimensional space. Then we will argue the topological change with respect to the fundamental group on this problem instead of knot invariance as in [30].

The organization of this paper is as follows. Section 2 reviews the classical shape of an elastica in two-dimensional flat space adding the infinity point, i.e. $\mathbb{C}P^1$. In section 3, we investigate the statistical mechanics of an elastica. First, we consider the thermal fluctuation of the extremum point of a partition function of the elastica in terms of the path integral method. Then we obtain the MKdV hierarchy by physical requirements. Second, we investigate the moduli space of the quasi-classical elastica, which is the extremum point of the partition function. Finally, we sum the Boltzmann weight over the moduli space and obtain an exact formulation of the partition function. In section 4, we discuss the obtained partition function and comment upon the relation between the Goldstein–Petrich method and the Becchi–Rouet–Stora (BRS) quantization of the gauge field [31] and a critical point of this mode.

2. Classical shape of elastica

Here we review a shape of a closed elastica in two-dimensional space [10–12, 32]. We denote by C a shape of the elastica embedded in the projective complex line (or the Riemann sphere) $\mathbb{C}P^1$ and by $X(s)$ its affine vector

$$S^1 \ni s \mapsto X(s) \in C \subset \mathbb{C}P^1 \qquad X(s+L) = X(s) \tag{2.1}$$

where L is the length of the elastica. The Frenet–Serret relation will be expressed as

$$\exp(\mathrm{i}\phi) = \partial_s X \qquad \partial_s \exp(\mathrm{i}\phi) = \mathrm{i}k\exp(\mathrm{i}\phi) \tag{2.2}$$

where ϕ is a real-valued function of s and k is the curvature of the curve C, $k := \partial_s\phi$: $\phi(s+L) = \phi(s)$ and $k(s+L) = k(s)$. Here we have chosen the metric of the curve as the induced metric from the natural metric of $\mathbb{C} \subset \mathbb{C}P^1$; by this choice ϕ is real valued.

As Daniel Bernoulli suggested to Euler [1], the energy integral of the elastica is given as

$$E = \int_0^L \mathrm{d}s\, k^2 \tag{2.3}$$

and the shape of the elastica is realized as its stationary point. Here we assume that the elastica does not stretch and preserves its infinitesimal length; since deformation of the elastica is regarded as one parameter transformation, the assumption implies that the transformation is isometric [10–13, 32]. Thus we refer to this condition as the isometry condition.

Following the minimal principle, we derive the differential equation. We consider the variation of the curve $C \to C_\varepsilon$ under the isometry condition [8–11]

$$X \to X_\varepsilon = X + \varepsilon(U_1 + \mathrm{i}U_2)\exp(\mathrm{i}\phi) \qquad U_1(L) = U_1(0) \qquad U_2(L) = U_2(0) \tag{2.4}$$

where εU_1 and εU_2 are infinitesimal real-valued functions.

Since the infinitesimal length of the curve is given as

$$\mathrm{d}s^2 = \mathrm{d}\bar{X}\,\mathrm{d}X = \partial_s\bar{X}\partial_s X\,\mathrm{d}s^2 \tag{2.5}$$

the corresponding length of the C_ε

$$\begin{aligned}\mathrm{d}\bar{X}_\varepsilon\,\mathrm{d}X_\varepsilon &= (1 + \varepsilon((\partial_s - \mathrm{i}k)U_1 - \mathrm{i}(\partial_s - \mathrm{i}k)U_2))(1 + \varepsilon((\partial_s + \mathrm{i}k)U_1 + \mathrm{i}(\partial_s + \mathrm{i}k)U_2))\,\mathrm{d}s^2 \\ &= (1 + 2\varepsilon(\partial_s U_1 - kU_2))\,\mathrm{d}s^2 + \mathcal{O}(\varepsilon^2)\end{aligned} \tag{2.6}$$

must be $\mathrm{d}s^2$ modulo ε^2 due to the isometry condition. Hence we obtain the relation

$$\partial_s U_1 = kU_2. \tag{2.7}$$

The tangential angle of C_ε is given as

$$\begin{aligned}\partial_\varepsilon &= \frac{1}{\mathrm{i}}\log\partial_s X_\varepsilon \\ &= \phi + \varepsilon(k + \partial_s k^{-1}\partial_s)U_1.\end{aligned} \tag{2.8}$$

Its curvature is expressed as

$$\begin{aligned}k_\varepsilon &= \exp(-\mathrm{i}\phi_\varepsilon)\partial_s^2 X_\varepsilon \\ &= k + \varepsilon\partial_s((k + \partial_s k^{-1}\partial_s)U_1).\end{aligned} \tag{2.9}$$

Finally, we obtain the variation of the energy functional

$$\int_0^L \mathrm{d}s\, k_\varepsilon^2 = \int_0^L \mathrm{d}s\left(k^2 + \varepsilon\left(\frac{1}{2}\partial_s(k^2) + \partial_s\frac{\partial_s^2 k}{k}\right)U_1\right) + \mathcal{O}(\varepsilon^2). \tag{2.10}$$

From the variational equation

$$\frac{\delta E[k_\varepsilon]}{\delta(\varepsilon U_1)} = 0 \tag{2.11}$$

the nonlinear differential equation is given as the equation of the shape of the static elastica

$$\partial_s \left(\frac{1}{2}k^2 + \frac{\partial_s^2 k}{k}\right) = 0 \tag{2.12a}$$

and thus

$$a_1 k + \tfrac{1}{2}k^3 + \partial_s^2 k = 0 \tag{2.12b}$$

where a_1 is the integral constant. This equation is known as the static MKdV equation in soliton theory if we derivate it by s again.

First, we note that (2.12a, b) are also equations of the energy functional [10, 11]

$$E = \int_0^L \mathrm{d}s (k^2 + A_1 \cos\phi + A_2) \tag{2.13}$$

where the second term means the constraint for the relative position of $X(0) - X(L)$ and the third one is for the total length L [7, 10, 11, 13]. The sufficiency of the third term is trivial. From (2.9), the second term becomes

$$A_1 \int \mathrm{d}s \cos(\phi_\varepsilon) = A_1 \int \mathrm{d}s \left(\cos(\phi) - \varepsilon\left(kU_1 + \partial_s \frac{\partial_s U_1}{k}\right)\sin(\phi)\right) + \mathcal{O}(\varepsilon^2) \tag{2.14}$$

and by partial integration, the second term on the right-hand side vanishes for any U_1. This vanishing occurs owing to the compatibility between the MKdV equation and the static sine–Gordon equation

$$\partial_s^2 \phi + A_1 \sin\phi = 0. \tag{2.15}$$

(2.15) comes from the natural variation of ϕ of (2.13) [1, 4, 7, 10, 11, 13]. Hence (2.12a, b) can be also regarded as the stationary point of (2.13). In fact it is known that solutions of (2.12a, b) are in agreement with those of (2.15) as will be shown later. It should also be noted that (2.13) is the σ-model with the topological term and was essentially discovered in the 18th century. In other words, the system of the elastica can be regarded as a SO(2) principal bundle over S^1 and the cosine term in (2.13) is a local version of the fundamental group $\pi_1(S^1) = \mathbb{Z}$ [6].

Solutions of (2.12a, b) are completely expressed by the elliptic functions. Multiplying $\partial_s k$ and integrating s [4, 11, 13, 14, 32, 33], we obtain the relation

$$(\partial_s k)^2 = -\tfrac{1}{4}(k^4 - a_1 k^2 + a_2). \tag{2.16}$$

Introducing the quantities, $\beta_2 - \beta_2 := a_1$, $l = \sqrt{\beta_1/(\beta_1 + \beta_2)}$, and $\beta_3 := \sqrt{\beta_1/\beta_2}/2$, k is expressed by the Jacobi elliptic function [34]

$$k = \sqrt{\beta_1}\, \mathrm{cn}(\beta_3 s, l). \tag{2.17}$$

Transformation from the solutions of (2.12a, b) to those of (2.15) is given by the identities of the integrand in the elliptic integral between trigonometric function and polynomial expressions [34]. After integrating the differential equations, we obtain [1, 4, 10, 11, 13, 33]

$$X(s) = \frac{2}{\beta_3} E(\mathrm{am}(\beta_3 s), l) - s - \mathrm{i}\frac{2l}{\beta_3}(\mathrm{cn}(\beta_3 s) - 1) \tag{2.18}$$

where $E(\cdot, l)$ is the incomplete elliptic integral of the second kind and am is the Jacobi amplitude function [34]. Due to the closed condition

$$X(0) = X(L) \tag{2.19}$$

there is an eight-figure shape [1, 4, 11, 13, 16, 33]; the modulus of the elliptic function is $l = 0.908\,909\ldots$ and the ratio of the fundamental parameters is $K'/K = 0.709\,46\ldots.$

Thus, in the set of solutions of (2.12a, b), there are only two types of closed elastica in $\mathbb{C}$ up to the translation of their centroid, global rotation and scaling; circle $k = 2\pi/L$ and eight-figure shape. Here, though we have chosen solutions such as $k = 2\pi/L$, there also exist other solutions like $\{k = 2\pi n/L \mid n \geqslant 1\}$.

On the other hand, by taking the limit $L \to \infty$ and by considering ones in $\mathbb{C}P^1$, more kinds of closed elastica in $\mathbb{C}P^1$ are allowed. These solutions were classified by Euler in the 18th century [1, 4].

In this paper the set of shapes of elastica obtained as solutions of (2.12a, b) will be denoted as $\mathfrak{S}_{\rm cls}$

$$\mathfrak{S}_{\rm cls} = \{C \mid C \text{ is a solution of } (2, 12a, b)\} \tag{2.20}$$

and the energy functional is expressed by

$$E_{\rm cls}[C] = \int_0^L {\rm d}s\, k^2 \qquad C \in \mathfrak{S}_{\rm cls}. \tag{2.21}$$

3. Statistical mechanics of elastica

In this section, we consider the statistical mechanics of a closed elastica and investigate its behaviour in a heat bath. We continue to allow the crossing of the elastica even in two-dimensional space. It is set up so that there are many independent laboratory dishes in which large polymers like DNA individually exist one by one. A looped elastica is in the liquid whose temperature is determined and whose viscosity is very large. The liquid is a kind of heat bath. Then the kinetic energy of the elastica is suppressed in equilibrium state due to dissipation but owing to the fluctuation by temperature noise, the elastica sometimes jumps from a quasi-stable state to other quasi-stable states.

The partition function of the elastica is given as in [24]

$$Z = \int DX \exp\left(-\beta \int_0^L {\rm d}s\, [k^2]\right). \tag{3.1}$$

Here we prohibit a change of the local length of the elastica, the isometry condition being maintained. In [24], it was written that if one also dealt with the kinematic term, it would be decoupled with the potential term (2.3). However, as we employ the isometry condition, this statement cannot be guaranteed because the kinetic term is also restricted by the isometry condition [8, 9] and is strongly coupled with the shape of the elastica.

3.1. Quasi-classical motion

By the quasi-classical method in path integration [35, 36], we evaluate the partition function (3.1). We expand the affine vector around an extremum point of the integral

$$X = X_{\rm qcl} - \epsilon(u_1(s) + {\rm i}u_2(s))\exp({\rm i}\phi_{\rm qcl}) + \mathcal{O}(\epsilon^2) \tag{3.2}$$

where ϵ is an infinitesimal parameter, $\epsilon \propto 1/\sqrt{\beta}$ and $\phi_{\rm qcl}$ is the tangent angle of the quasi-classical curve of elastica. $X_{\rm qcl}$ is an affine vector of the extremum point of the functional integral and will be determined later. In the path integral, terms with higher orders of ϵ also play an important role and thus we must pay attention to the higher perturbations of ϵ here. Hence we will assume that X is parametrized by a parameter t and that the difference between X and $X_{\rm qcl}$ can be expressed by

$$X(s,t) := {\rm e}^{\epsilon\partial_t} X_{\rm qcl}(s,t) \qquad \epsilon\partial_t X_{\rm qcl} = X - X_{\rm qcl} + \mathcal{O}(\epsilon^2). \tag{3.3}$$

Since for an analytic function $f(x)$, $\mathrm{e}^{a\partial_x} f(x) = f(x+a)$, $X(s,t)$ can be expressed as $X(s,t) = X_{\rm qcl}(s, t+\epsilon)$, where the direction t differs from that of s; the domain of functional integration (3.1) deviates from the domain S^1 of the classical map (2.1). Then (3.2) becomes

$$-\partial_t X_{\rm qcl} = (u_1 + \mathrm{i}u_2)\exp(\mathrm{i}\phi_{\rm qcl}) \qquad u_1(L) = u_1(0) \qquad u_2(L) = u_2(0). \tag{3.4}$$

This is virtual dynamics of the curve [8–10]. As well as the argument in section 2, due to the isometry condition, we require $[\partial_t, \partial_s] = 0$ for X. Then the isometry condition exactly preserves, $\mathrm{d}s \equiv \mathrm{d}s_{\rm qcl}$ by the definition (3.3). This isometry relation should be compared with (2.6) which is isometry modulo ϵ^2. It should also be noted that although ϵ is constant, dependence of the variation upon the position s is given though (3.4) and $u_a(s)$, $a = 1, 2$. Thus (3.3) is not a trivial deformation in general.

We have the relation

$$-\partial_t \exp(\mathrm{i}\phi_{\rm qcl}) = ((\partial_s u_1 - u_2 k_{\rm qcl}) + \mathrm{i}(\partial_s u_2 + u_1 k))\exp(\mathrm{i}\phi_{\rm qcl}). \tag{3.5}$$

Noting that ϕ and u are real valued, we obtain (2.7) again from the first term and solve the differential equation between u_1 and u_2 [8, 10]

$$\partial_s u_1 = k_{\rm qcl} u_2 \qquad u_1 = \int^s \mathrm{d}s\, u_2 k_{\rm qcl} =: \partial_s^{-1} u_2 k_{\rm qcl}. \tag{3.6}$$

Here ∂_s^{-1} has a parameter $c \in \mathbb{R}$ as an integral constant and coincides with the pseudo-differential operator.

Then (3.5) is reduced to the equations

$$\partial_t k_{\rm qcl} = -\Omega u_2 \qquad \Omega := \partial_s^2 + \partial_s k_{\rm qcl}\partial_s^{-1} k_{\rm qcl}. \tag{3.7}$$

From (3.3) and $[\partial_t, \partial_s] = 0$ for X, ϕ is calculated as

$$\phi(s,t) = \phi_{\rm qcl}(s, t+\varepsilon) = \mathrm{e}^{\epsilon\partial_t}\phi_{\rm qcl}(s,t) = \phi_{\rm qcl} + \epsilon\partial_t\phi_{\rm qcl} + \frac{1}{2!}\epsilon^2\partial_t^2\phi_{\rm qcl} + \cdots. \tag{3.8}$$

Then noting $k^2(s,t) = k^2_{\rm qcl}(s, t+\epsilon)$, the energy functional is expressed as

$$\begin{aligned}\int k^2\,\mathrm{d}s &= \int (k_{\rm qcl}^2 + 2\epsilon k_{\rm qcl}\partial_t k_{\rm qcl} + \epsilon^2((\partial_t k_{\rm qcl})^2 + k_{\rm qcl}\partial_t^2 k_{\rm qcl}) + \cdots)\,\mathrm{d}s \\ &= \int (k_{\rm qcl}^2 - 2\epsilon k_{\rm qcl}\Omega u_2 + \epsilon^2((\partial_t k_{\rm qcl})^2 + k_{\rm qcl}\partial_t^2 k_{\rm qcl}) + \cdots)\,\mathrm{d}s \\ &=: E_{\rm qcl} + \delta^{(1)}E + \delta^{(2)}E + \cdots. \end{aligned} \tag{3.9}$$

Using (3.6), if we perform the functional derivative of E in u_1, we obtain the classical equations (2.12a, b) again.

Since the quasi-classical configuration is realized as the extremum point of the functional space, we must impose the relation

$$\delta^{(1)}E = 0. \tag{3.10}$$

According to the philosophy of this method, we must sum the weight function over all extremum points. Since they are extremum rather than stationary points, they need not be realized as classical equations like (2.12a, b).

It has been assumed that the deformation is characterized by one parameter t. However, in the mode analysis of a linear deformation system, the coefficients of each mode govern the classification and the magnitude of the deformation. When one takes notice of one of the modes and its coefficient as a parameter of the system like t of this argument, we can consider a set of the modes and their coefficients and sum them. Although the elastica system is not linear, there is no requirement that we must use only one parameter t to describe this system. In the sense of the statistical mechanics (and also quantum mechanics) one must sum the weight function over events if the possibility of occurrence of the events can be considerable regardless of the magnitude of its contribution; when it does not affect the total system enough, the degree of affection is built into the weight function. Thus, one must search for all possible extremum points as a duty if one wishes to calculate a partition function.

Furthermore, in the path integral method, the entropy as a statistical effect appears as a volume of the function space; in a microcanonical system at energy E_0, the entropy S of the system is defined as $S := \log Z|_{E=E_0}$ and can be regarded as the logarithm of the volume of the function space. From primitive consideration, the dimension of the volume of the functional space in the statistical physics is related to the degrees of freedom corresponding to E_0.

Since the deformation parameter determines the functional space of this system, as will be shown later, and the degree of freedom of the elastica is not finite, its dimension need not be one. In other words, the search for the extremum point (3.12) must take precedence over paying attention to the dimension of the parameter space.

Noting the relation (3.6) and the above arguments, if Ωu_2 could be regarded as another function u_2' of the variation of the normal direction in (3.2), we might find the relation

$$\int \mathrm{d}s\, k_{\mathrm{qcl}} \Omega u_2 \sim \int \mathrm{d}s\, k_{\mathrm{qcl}} u_2' = \int \mathrm{d}s\, \partial_s u_1' = 0. \tag{3.11}$$

Accordingly, we introduce the sequence for mathematical times $\boldsymbol{t} := (t_1, t_3, t_5, \ldots, t_{2n+1}, \ldots)$ so that (3.11) is satisfied. We redefine the fluctuation (3.2) and introduce an infinite parameter family, which can sometimes become a finite set as we show later

$$X = \mathrm{e}^{(1/\sqrt{\beta}) \sum_{n=0} \delta t_{2n+1} \partial_{t_{2n+1}}} X_{\mathrm{qcl}} = X_{\mathrm{qcl}} + (1/\sqrt{\beta}) \sum_{n=0} \delta t_{2n+1} \partial_{t_{2n+1}} X_{\mathrm{qcl}} + \mathcal{O}(1/\beta). \tag{3.12}$$

Here ϵ was replaced with $(1/\sqrt{\beta})\delta t_{2n+1}$ and $\partial_{t_{2n+1}} X_{\mathrm{qcl}}$ is expressed as

$$-\partial_{t_{2n+1}} X_{\mathrm{qcl}} = (u_1^{(n)} + \mathrm{i} u_2^{(n)}) \exp(\phi_{\mathrm{qcl}}) \qquad u_1^{(n)} = \partial_s^{-1} k_{\mathrm{qcl}} u_2^{(n)} \qquad u_2^{(n)} = \Omega^n u_2^{(0)} \tag{3.13}$$

with integral constants c vanishing for $n > 1$. $u_2^{(0)}$ is an appropriate function of s.

Then the variation of the energy functional is calculated as

$$\begin{aligned}\int k^2\, \mathrm{d}s &= \int \left(k_{\mathrm{qcl}}^2 + (2/\sqrt{\beta}) \sum_n \delta t_{2n+1} k_{\mathrm{qcl}} \partial t_{2n+1} k_{\mathrm{qcl}} \right) \mathrm{d}s + \mathcal{O}(1/\beta) \\ &= \int \left(k_{\mathrm{qcl}}^2 - (2/\sqrt{\beta}) \sum_n \delta t_{2n+1} k_{\mathrm{qcl}} \Omega u_2^{(n)} \right) \mathrm{d}s + \mathcal{O}(1/\beta) \\ &= \int \mathrm{d}s\, k_{\mathrm{qcl}}^2 - (2/\sqrt{\beta}) \sum_n \delta t_{2n+1} \int \mathrm{d}s\, \partial_s u_1^{(n+1)} + \mathcal{O}(1/\beta) \\ &= \int \mathrm{d}s\, k_{\mathrm{qcl}}^2 + \mathcal{O}(1/\beta). \end{aligned} \tag{3.14}$$

Thus, for the variations along the directions, the energy of the system is an invariant modulo $1/\beta$. Without work, we can move it for these directions $\delta t_{2n+1} \partial_{2n+1} X$ [10, 37, 38].

However, for this sequence, the infinite differential equations appear [8, 9]

$$\partial t_{2n+1} k_{\mathrm{qcl}} = -\Omega^n u_2^{(0)}. \tag{3.15}$$

The recursion equations (3.13) are determined by the initial condition $u_2^{(0)}$. By following the theory of the quasi-classical method of the path integral, this sequence must contain the classical equations (2.12a, b). On the other hand, for a close elastica, there is a trivial continuous symmetry which is the translation of the coordinate system s along the curve C. Hence (3.15) should also include such translation symmetry. Of course, they contain other equations as quasi-stable shapes as shown later.

Although there might be other choices, we select minimal subspaces of the functional space in order to satisfy the above requirements. This choice will be justified later. As we performed the variational computation in section 2, we choose the initial state as

$$u_2^{(0)} = 0 \qquad u_2^{(n)} = \Omega^n \partial_s k_{\rm qcl} \qquad \text{for } n \geqslant 1. \tag{3.16}$$

Then the minimal set of the virtual equations of motion, which satisfy the physical requirements, is

$$\partial_{t_{2n+1}} k_{\rm qcl} = -\Omega^n \partial_s k_{\rm qcl} \qquad \partial_{t_{2n+3}} k_{\rm qcl} = \Omega \partial_{2n+1} k_{\rm qcl}. \tag{3.17}$$

First, several equations and u are given as follows

$$u_1^{(0)} = 1 \qquad u_2^{(0)} = 0 \tag{3.18a}$$

$$u_1^{(1)} = \tfrac{1}{2} k_{\rm qcl}^2 \qquad u_2^{(1)} = \partial_s k_{\rm qcl} \tag{3.18b}$$

$$u_1^{(2)} = \tfrac{3}{8} k_{\rm qcl}^4 - \tfrac{1}{2}(\partial_s k_{\rm qcl})^2 + k \partial_s^2 k_{\rm qcl} \qquad u_2^{(2)} = \tfrac{3}{2} k_{\rm qcl}^2 \partial_s k_{\rm qcl} + \partial_s^3 k_{\rm qcl} \tag{3.18c}$$

$$n = 0: \qquad \partial_{t_1} k_{\rm qcl} + \partial_s k_{\rm qcl} = 0 \tag{3.19a}$$

$$n = 1: \qquad \partial_{t_3} k_{\rm qcl} + \partial_s^3 k_{\rm qcl} + \tfrac{3}{2} k_{\rm qcl}^2 \partial_s k_{\rm qcl} = 0 \tag{3.19b}$$

$$n = 2: \qquad \partial_{t_5} k_{\rm qcl} + \partial_s^5 k_{\rm qcl} + \tfrac{15}{8} k_{\rm qcl}^4 \partial_s k_{\rm qcl} + \tfrac{5}{2} (\partial_s k_{\rm qcl})^3 + \tfrac{5}{2} k_{\rm qcl}^2 \partial_s^3 k_{\rm qcl} + 10 k_{\rm qcl} \partial_s k_{\rm qcl} \partial_s^2 k_{\rm qcl} = 0. \tag{3.19c}$$

Since Ω is identified with the Gel'fand–Dikii operator for the MKdV equation, (3.17) is regarded as the MKdV hierarchy and $u_1^{(n)}$ are Hamiltonians of the MKdV hierarchy [8–10, 39].

Next, we consider the solutions of these equations. (3.19a) means the freedom of choice of the origin of s and has only a trivial solution $k(s - t_1)$; t_1 is the origin of s. Hence (3.16) contains the trivial symmetry. Since we are concerned with nontrivial deformation, we must consider only the properties of (3.17) for $n \geqslant 1$. The derivative of (2.12b) can be described as

$$c \partial_s k = \Omega \partial_s k. \tag{3.20}$$

By $t_3 = s/c$, this is identified with $n = 1$ in equation (3.19b). Here $\partial_s k$ of the solutions of (2.12b) are interpreted as the eigenvectors of Ω and c is an eigenvalue of the operator Ω. Thus (3.17) becomes

$$\partial_{t_{2n+1}} k = -\Omega^n \partial_s k = -c^n \partial_s k. \tag{3.21}$$

Hence the solutions of (2.12a, b) can be solutions of all the equations in (3.17). In fact, for a stable solution $k =$ constant, all $u_2^{(n)} \equiv 0$ and thus all equations in (3.17) are satisfied.

It should be noted that (3.21) arises from the fact that $u_1^{(m)}$ agree with the Hamiltonians of the MKdV hierarchy and are regarded as conservation quantities for nth equations of $n < m$ [39]. Hence using soliton theory, any solutions of (3.19b) are solutions of higher equations in (3.17) $n \geqslant 2$. Due to the requirement of the quasi-classical solutions, any solutions must satisfy all of nth equations ($n \geqslant 1$) in (3.17). Hence, we may deal with only the solutions of the MKdV equation (3.19b) as the quasi-classical solutions of this system.

Then the sequence (3.17) fulfils the physical requirements. In other words, the solution space of the MKdV equation (3.19b) is the minimal space containing the classical solutions and translation symmetry and fulfils the quasi-stable condition (3.10).

Since for the variations along the directions $\boldsymbol{t}$ of (3.17), the energy of the system (3.14) is invariant, the fluctuation of the quasi-classical shape $k_{\rm qcl}(s, t_{2n+1})$ should be regarded as (generalized) Jacobi fields of the Goldstone mode [37, 38] even though they do not obey a linear differential equation in general.

Here we remark that the MKdV hierarchy naturally appears by physical requirements in the functional integral. It is very surprising because the MKdV hierarchy has infinite Hamiltonians and time axes; in classical kinetic theory, these quantities cannot be physically interpreted. Furthermore, it should be contrasted with the works related to space curves in [8, 9, 33], in which the authors chose (3.16) by hand without any physical requirement.

Due to the fluctuation of the heat noise, the equation of the elastica in a heat bath obeys the MKdV equation (3.19*b*) rather than the static MKdV equation (2.12*a*, *b*). Let the set of solutions referred as

$$\mathfrak{S}_{\rm qcl} = \{C \mid C \text{ is a solution of } (3.19b)\}. \tag{3.22}$$

Of course $\mathfrak{S}_{\rm qcl}$ contains $\mathfrak{S}_{\rm cls}$. In $\mathfrak{S}_{\rm qcl}$, various shapes appear as the quasi-classical solutions in a heat bath. For example, there should exist a deformed circle as a solution of (3.19*b*) because (3.19*b*) is of an initial value problem. As another example, there are other topological solutions of the different sectors of the fundamental group

$$\frac{1}{2\pi}\int \mathrm{d}\phi_{\rm qcl} \in \mathbb{Z}. \tag{3.23}$$

3.2. Moduli of closed quasi-classical elastica

Here we go back to compute the partition function (3.1), which was formally defined. First, the problem arises how (3.1) should be interpreted. According to the philosophy of the canonical ensemble, the partition function should be calculated by the sum over all distinguishable and possible curves satisfying the isometry condition with Boltzmann weight. In the calculus, a different topological class of (3.23) will be summed over.

However, the partition function (3.1) naturally diverges because the energy function is invariant for the affine transformation, i.e. for the translation and the global rotation. Fixing C, if we denote X as

$$X(s) = X_g + X_r(s) \qquad \int \mathrm{d}s\, X_r(s) = 0 \tag{3.24}$$

where X_g is the centroid of the curve C, then the measure of the partition function can be rewritten as

$$\int \mathrm{D}X\, \mathrm{e}^{-\beta E} = \int \mathrm{D}X_g \int \mathrm{D}X_r(C)\, \mathrm{e}^{-\beta E(C)}. \tag{3.25}$$

The $\int \mathrm{D}X_g$ is the volume of the base space $\mathbb{C}P^1$ and diverges. Including the rotational symmetry and the inner translation symmetry $s \to s - t_1$, we redefine the partition function which is divided by the volume of the affine transformation of the base space and length of the elastica

$$Z_{\rm reg} := \frac{\int \mathrm{D}X\, \mathrm{e}^{-\beta E}}{\mathrm{vol}(\mathrm{Aff}(\mathbb{C}P^1))L}. \tag{3.26}$$

Then we concentrate on the shape of the elastica. We must classify the shape of the elastica and sum over all possible shapes. In other words, we must investigate the moduli space of the quasi-classical elastica

$$\mathfrak{M}_{\rm qcl} := \frac{\mathfrak{S}_{\rm qcl}}{\mathrm{Aff}(\mathbb{C}P^1) \times S^1}. \tag{3.27}$$

First, we consider the moduli space of the MKdV equation. The moduli of the MKdV equation was investigated as the KP-hierarchy using Sato theory [40–42]. (By the Miura map, the MKdV hierarchy is transformed to the KdV hierarchy and the KdV hierarchy is a subset of the KP hierarchy [41].) The moduli of the MKdV equation is classified with the genus $g \in \mathbb{N}$ of the hyperelliptic Reimannian surface (hyperelliptic curve) R_g, which is the finite gap energy manifold (Bloch band spectrum) of the wavefunctions in the inverse scattering method of the MKdV equation [40–46]. Hence the moduli of the closed elastica is also classified by the genus

$$\mathfrak{M}_{\text{qcl}} = \coprod_g \mathfrak{M}_{\text{qcl}}^{(g)}. \tag{3.28}$$

We will call the genus of the MKdV equation with the boundary condition genus of the elastica g. In fact, the classical solutions of (2.12a, b) and (3.20) correspond to the shapes of elastica of genus zero and one, because these energy manifolds appearing in its inverse scattering method exhibit a Riemannian sphere and an elliptic curve, respectively [43]. (It should be noted that even in the quasi-classical equation, the solutions of the circle and the eight-figure are unique up to homothety (similarity transformation) for $g = 0$ and $g = 1$, respectively.) Using our knowledge of the properties of the universal Grassmannian manifold (UGM), we will consider the moduli of the closed elastica.

First, we consider the simplest case ($g = 0$), a circle $k = 2\pi n/L$, $n \geqslant 1$. In (3.9), the quasi-classical action of these circles is expressed as

$$E_{\text{qcl}}[C_n] = \frac{2\pi n^2}{L} \qquad k(C_n) = \frac{2\pi n}{L}. \tag{3.29}$$

Hence for large n, the Boltzmann weight $\exp(-\beta E_{\text{qcl}}[C_n])$ rapidly decreases. This situation is preserved for the elastica of higher genus.

Next, we consider elastica of genus one and answer the question why the number of the genus one solutions of closed elastica is only one, i.e., eight-figure shape, up to scaling. The moduli of the compact Riemannian surface of genus one (or elliptic curve) is conventionally expressed as $(1, \tau)$, $\tau \in \tilde{\mathfrak{M}}_{R_1}$

$$\begin{aligned} &\tilde{\mathfrak{M}}_{R_1} = H_+/\text{PSL}(2, \mathbb{Z}) \\ &H_+ := \{m \in \mathbb{C} \mid \text{Im}\,(m) \geqslant 0\} \\ &\text{PSL}(2, \mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - bc = 1, a, b, c, d \in \mathbb{Z} \right\} \Big/ (\pm 1). \end{aligned} \tag{3.30}$$

However, there is a dilatation freedom ($\tilde{K}, \tilde{K}' := \tilde{K}\tau$) and thus we will denote $\mathfrak{M}_{R_1} := \mathbb{R}_{>0} \times \tilde{\mathfrak{M}}_{R_1}$ to include the freedom: $\tilde{K} \in \mathbb{R}_{>0} := \{x \in \mathbb{R} \mid x > 0\}$. The Jacobi variety of an elliptic curve is given as $J_{1,\boldsymbol{m}} := \mathbb{C}/(\tilde{K}\mathbb{Z} \oplus \tilde{K}'\mathbb{Z})$ for $\boldsymbol{m} := (\tilde{K}, \tilde{K}')$. Since ϕ_{qcl} is a real analytic function of $s \in S^1 = \mathbb{R}/L\mathbb{Z}$, its domain embedded in $J_{1,\boldsymbol{m}}$ must be real. Thus only one-dimensional parametrization of ϕ_{qcl}, $S^1 \subset J_{1,\boldsymbol{m}}$, is allowed, which is a direct line in $J_{1,\boldsymbol{m}}$ and passes its origin, because $J_{1,\boldsymbol{m}}$ is a complex one-dimensional manifold. Since the moduli was divided by $\text{PSL}(2, \mathbb{Z})$, there are $\text{PSL}(2, \mathbb{Z})$ choices how such S^1 is embedded $J_{1,\boldsymbol{m}}$. We choose a function with period L/n, $n \in \mathbb{N} := \{n \in \mathbb{Z} \mid n \geqslant 1\}$, $\phi_{\text{qcl}}(s + L/n) = \phi_{\text{qcl}}(s)$. By the periodicity of $\phi_{\text{qcl}}(s)$, we fix $\tilde{K}$ for each embedding of S^1 into $J_{1,\boldsymbol{m}}$, then the moduli of the MKdV equation with period L is $\mathfrak{M}_{R_1} \times \mathbb{N} \times \text{PSL}(2, \mathbb{Z}) \times \mathbb{R}_{>0}/\mathbb{R}_{>0}$, which is equivalent with $\mathbb{N} \times H_+$.

On the other hand, the closed condition (2.19) in $\mathbb{C}P^1$ restricts the moduli of the elastica. We introduce a real analytic map

$$f_1 : \frac{\mathfrak{M}_{R_1} \times \mathbb{N} \times \text{PSL}(2, \mathbb{Z}) \times \mathbb{R}_{>0}}{\mathbb{R}_{>0}} \approx \mathbb{N} \times H_+ \to \mathbb{C}P^1$$

$$f_1(m) = X(L) - X(0). \tag{3.31}$$

Both H_+ and $\mathbb{C}P^1$ are complex one-dimensional spaces. The moduli of closed elastica with genus one is given as the inverse image of the zero point of f_1

$$\mathfrak{M}^{(1)}_{\rm qcl} = f_1^{-1}(0). \tag{3.32}$$

Due to the analyticity of the map f_1, $\mathfrak{M}^{(1)}_{\rm qcl}$ is a zero-dimensional manifold. Thus the kind of shapes are countable and due to the uniformity, there is only one solution for each $n \in \mathbb{N}$.

In ordinary computations [4, 10–14, 16, 32, 33], by reparametrizing S^1 as $\mathbb{R}/(L/n)\mathbb{Z}$, one starts with $\mathbb{C}/((L/n)\mathbb{Z} \oplus K'\mathbb{Z})$, $n \in \mathbb{N}$ and $K' \in H_+$ without dividing $\mathrm{PSL}(2,\mathbb{Z})$ and searches for the point satisfying the closed condition (2.19).

Similarly properties of the moduli of the closed elastica with genus $g(>1)$ will be investigated. It is well known that by the Sato theory, a characteristic of the KdV hierarchy in the KP hierarchy is to characterize its energy manifold in the inverse scattering method as a hyperelliptic curve in general (compact) Riemannian surfaces [40–42]. The Miura maps from the MKdV hierarchy to the KdV hierarchy are bijective. Thus we deal only with the hyperelliptic curves in this paper. First, we denote the moduli of the hyperelliptic curve of $g(>1)$ as $\tilde{\mathfrak{M}}_{R_g}$. Its element is conventionally expressed as (I_g, T_g), where I_g and T_g are $g \times g$ matrices; $T_g = (\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_g) = (\tau_{ij})$ and $I_g = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_g)$ is the unit matrix. As we did in the $g = 1$ case, we will deal with $\tilde{K}(I_g, T_g)$, $\tilde{K} \in \mathbb{R}_{>0}$ rather than (I_g, T_g) itself. It is known that the dimension of the moduli of the hyperelliptic curves, $\tilde{\mathfrak{M}}_{R_g}$, is $2g - 1$. Then we will also introduce a real $2g$-dimensional lattice for a point of the moduli $\boldsymbol{m} \in \mathfrak{M}_{R_g} := \mathbb{R}_{>0} \times \tilde{\mathfrak{M}}_{R_g}$ [44–46]

$$\Gamma_{\boldsymbol{m}} = \left\{ \sum_{j=1}^{g} m_j \tilde{K} \boldsymbol{e}_j + \sum_{j=1}^{g} n_j \tilde{K} \boldsymbol{\tau}_j \mid (m_1, m_j \in \mathbb{Z}) \right\} \tag{3.33}$$

and the Jacobi variety $J_{g,\boldsymbol{m}} := \mathbb{C}^g/\Gamma_{\boldsymbol{m}}$. If we determine a point $\boldsymbol{m}$ in $\mathfrak{M}_{R_g}$, we can uniquely construct the Jacobi variety, $J_{g,\boldsymbol{m}} := \mathbb{C}^g/\Gamma_{\boldsymbol{m}}$. From soliton theory, if the coordinates of $J_{g,\boldsymbol{m}}$ as a real manifold are expressed by $\boldsymbol{t}_{\rm KP} = (t_1, t_2, t_3, t_4, \ldots, t_{2_g})$ [40–42], its subset with odd indices $\boldsymbol{t}_g := (t_1, t_3, \ldots, t_{2g-1})$ can be identified with the part of $\boldsymbol{t}$ in (3.12). This identification can be guaranteed by the Krichever construction of the solution of the KP hierarchy [44] and Sato theory [40–42]. By the Krichever construction, it is known that each parametrization $t_n \in \boldsymbol{t}_{\rm KP}$ is a direct line passing the origin in the Jacobi variety. (3.17) and (3.19) are reduced to the linear differential equations in the Jacobi variety [44–46]. (Thus (3.17) can be recognized as the Jacobi equation of the Jacobi field of the system (3.14).)

Since the moduli of the hyperelliptic curves has also been divided by a discrete group $\mathrm{Sp}(g,\mathbb{Z})$ [44] like $\mathrm{PSL}(2,\mathbb{Z})$ of the $g = 1$ case, for fixing $\boldsymbol{m} \in \mathfrak{M}_{R_g}$, there are $N \times \mathrm{Sp}(g,\mathbb{Z})$ ways to embed S^1, as a period of ϕ, into $J_{g,\boldsymbol{m}}$. The number of ways are equivalent to the cardinal of $\mathbb{N} \times \mathrm{Sp}(g,\mathbb{Z})$ as a set. As we impose its periodicity of L/n ($n \in \mathbb{N}$) on S^1, the dilation parameter $\tilde{K}$ is determined. Let such moduli space be denoted as

$$\mathfrak{M}_{R_g,S^1} := \mathfrak{M}_{R_g} \times \mathbb{N} \times \mathrm{Sp}(g,\mathbb{Z})/\mathbb{R}_{>0}. \tag{3.34}$$

By choosing a point of the moduli space $\mathfrak{M}_{R_g,S^1}$, a $g \times g$ lattice $\Gamma_{\boldsymbol{m},S^1}$ is uniquely determined and the Jacobi variety is given as

$$\mathfrak{M}_{R_g,S^1} \to \{\Gamma_{\boldsymbol{m},S^1}\} \qquad J_{g,\boldsymbol{m},S^1} := \mathbb{C}^g/\Gamma_{\boldsymbol{m},S^1}. \tag{3.35}$$

We will fix a point of the modulus $\boldsymbol{m} \in \mathfrak{M}_{R_g,S^1}$ for a while. From the properties of the MKdV hierarchy, $\phi_{\rm qcl}$ is a real analytic function of $\boldsymbol{t}_g$. As we can expand it around a point

$\boldsymbol{t}_g$ using the properties of its real analyticity

$$\phi_{\rm qcl}(\boldsymbol{t}'_g) = \sum_{n_0,\dots,n_g} a_{n_0},\dots,n_1(t'_1 - t_1)^{n_0}(t'_3 - t_3)^{n_1}\dots(t'_{2g-1} - t_{2g-1})^{n_g} \qquad (3.36)$$

$\boldsymbol{t}_g$ must be a system of real parameters in $J_{g,\boldsymbol{m},S^1}$. By analytic continuation, $\boldsymbol{t}_g$ can be locally complexified. On the other hand, the Jacobi variety has a canonical complex structure

$$\mathcal{J} : J_{g,\boldsymbol{m},S^1} \to J_{g,\boldsymbol{m},S^1} \qquad \mathcal{J}^2 = -1 \qquad (3.37)$$

which consists of its affine (vector) structure. By the structure $\mathcal{J}$, there is a set of real g-dimensional submanifolds $\{\Sigma_{g,\boldsymbol{m},S^1}\}$ which includes the orbit of $s(\in S^1)$ as its one-dimensional submanifold. Due to the analyticity of $\phi_{\rm qcl}$ over $J_{g,\boldsymbol{m},S^1}$, this complexification of (3.36) cannot contradict with the complex structure $\mathcal{J}$. Then $\boldsymbol{t}_g$ can be regarded as an element of $\Sigma_{g,\boldsymbol{m},S^1}$. We will refer to such embedding as

$$\sigma_0 : \Sigma_{g,\boldsymbol{m},S^1} \hookrightarrow J_{g,\boldsymbol{m},S^1} \qquad (3.38)$$

and the set of σ_0 is expressed as $\mathrm{Em}_0(\Sigma_{g,\boldsymbol{m},S^1}, J_{g,\boldsymbol{m},S^1})$.

Then we can construct the fibre structure for $\mathrm{Em}_0(\Sigma_{g,\boldsymbol{m},S^1}, J_{g,\boldsymbol{m}})$, because for a way to such embedding, there is the trajectory space $\Sigma_{g,\boldsymbol{m},S^1}/S^1$ as fibre space, where $(t_3,\dots,t_{2g-1}) \in \Sigma_{g,\boldsymbol{m},S^1}/S^1$. We refer to the fibre bundle as $\mathfrak{I}_{\boldsymbol{m}}$

$$\begin{array}{ccc} \Sigma_{g,\boldsymbol{m},S^1}/S^1 & \longrightarrow & \mathfrak{I}_{\boldsymbol{m}} \\ & & \downarrow_{\pi_{\rm traj}} \\ & & \mathrm{Em}_0(\Sigma_{g,\boldsymbol{m},S^1}, J_{g,\boldsymbol{m},S^1}). \end{array} \qquad (3.39)$$

This fibre space also depends upon the point of the moduli space of the Riemannian surfaces $\mathfrak{M}_{R_g,S^1}$. Hence, the moduli of the periodic solutions of $\phi_{\rm qcl}(s,\boldsymbol{t}_g)$, which is written as $\mathfrak{M}^{(g)}_{\rm period}$, also has a fibre structure

$$\begin{array}{ccc} \mathfrak{I}_{\boldsymbol{m}} & \longrightarrow & \mathfrak{M}^{(g)}_{\rm period} \\ & & \downarrow_{\pi_{\rm period}} \\ & & \mathfrak{M}_{R_g,S^1}. \end{array} \qquad (3.40)$$

For each point $\boldsymbol{m} \in \mathfrak{M}_{R_g,S^1}$, the fibre bundle $\mathfrak{I}_{\boldsymbol{m}}$ (3.39) stands up as a fibre of $\mathfrak{M}^{(g)}_{\rm period}$.

By the closed condition, we must restrict the moduli space. Here we consider a real analytic map like (3.31)

$$f_g : \mathfrak{M}^{(g)}_{\rm period} \to \mathbb{C}P^1$$
$$f_g(\mu) = X(L) - X(0). \qquad (3.41)$$

Consequently, we obtain the moduli of the closed elastica, which is expressed as

$$\mathfrak{M}^{(g)}_{\rm qcl} = f_g^{-1}(0) \subset \mathfrak{M}^{(g)}_{\rm period}. \qquad (3.42)$$

Since the image of f_g is a real two-dimensional manifold, for $g > 2$, $\dim(\mathfrak{M}^{(g)}_{\rm qcl}) \geqslant 1$ and $\mathfrak{M}^{(g)}_{\rm qcl}$ is measurable.

For simplicity, we introduce the following notation

$$\mathfrak{M}^{(g)}_{t_m=0} := \pi_{\rm traj}(\mathfrak{M}^{(g)}_{\rm qcl})$$
$$\mathfrak{X}^{(g)} \ni n : \mathfrak{M}^{(g)}_{t_m=0} \to \mathfrak{M}^{(g)}_{t_m=0,n} \qquad \text{or} \qquad \mathfrak{M}^{(g)}_{t_m=0} = \oplus_{n\in\mathfrak{X}^{(g)}}\mathfrak{M}^{(g)}_{t_m=0,n}$$
$$\text{for} \qquad (n,\mathfrak{m}) \in \mathfrak{M}^{(g)}_{t_m=0} \qquad \tilde{\Sigma}_{\mathfrak{m},n} := \pi^{-1}_{\rm traj}(n,\mathfrak{m}) \qquad (3.43)$$

where $\mathfrak{X}^{(g)}$ is the countable part of $\mathfrak{M}^{(g)}_{\boldsymbol{t}_m=0}$ and $\mathfrak{M}^{(g)}_{\boldsymbol{t}_m=0,n}$, the restriction of $\mathfrak{M}^{(g)}_{\boldsymbol{t}_m=0}$ for a point $n \in \mathfrak{X}^{(g)}$, is the measurable part. Here $\tilde{\Sigma}_{\mathfrak{m},n}$ has coordinates $(t_3, \ldots, t_{2g-1}) =: \boldsymbol{t}_{\mathfrak{m}}$. Hence, there is a map from the moduli to the shape of the elastica

$$h : \mathfrak{M}^{(g)}_{\mathrm{qcl}} \ni (n, \mathfrak{m}, \boldsymbol{t}_{\mathfrak{m}}) \mapsto C^{(n)}_{\mathfrak{m}}(\boldsymbol{t}^{(n)}_{\mathfrak{m}}) \subset \mathbb{C}P^1. \tag{3.44}$$

For a perturbative deformation like distortion to an ellipse from a circle, the energy manifold in the inverse scattering method has infinite gaps (or genera) in general. Thus such deformation is expressed in the moduli of $g \to \infty$ and, due to the integrability of the MKdV equation, the deformation can be predicted like an harmonic oscillator around a stable point. This picture is supported by the linearized method of the nonlinear equation and is also built into the above argument of the limit $g \to \infty$.

3.3. Partition function

As we finish classifying the solution space formally, we consider the fluctuation of the elastica again. It should be noted that there is an upper limit of the sequence $u_2^{(n)}$ corresponding to the genus of the elastica. If for $n = N$,

$$\Omega u_2^{(N)} \equiv \lambda u_2^{(N)} \qquad \lambda \in \mathbb{R} \tag{3.45}$$

like (3.20), then $\delta t_{2(N+m)+1} \propto \delta t_{2N+1}$ for $m > 0$ because of (3.12) and

$$-\partial_{t_{2(N+m)+1}} u_1^{(N+m)} = k_{\mathrm{qcl}} u_2^{(N+m)} = k_{\mathrm{qcl}} \Omega^m u_2^{(N)} = \lambda^m \partial_{t_{2N+1}} u_1^{(N)}. \tag{3.46}$$

Accordingly there is no requirement for other fluctuation parameters $n > N$ because these fluctuation vectors are linearly dependent. The sequence of (3.12) should be truncated according to the philosophy of the canonical ensemble. Thus we denote such a minimal integer, which is a function of the solution, as

$$\mathrm{ind}_0 : C \to N(C) \in \mathbb{Z}. \tag{3.47}$$

However, from soliton theory [40–42] and the above argument, for $C \in \mathfrak{M}^{(g)}_{\mathrm{qcl}}$, we conclude that $\mathrm{ind}_0(C) = g$. Avoiding meaningless divergence, we will replace the infinite series in (3.12) with the finite sum from 1 to g depending upon the shape of elastica.

Since the direction of δt_1 is along the tangential direction of the elastica C_{qcl}, its effect has been treated as the integral of $\delta t_1 \propto s$ in (3.26). On the other hand, $\delta t_{2n+1} (n > 1)$ includes the normal direction fluctuation and we must integrate the Boltzmann weight over δt_{2n+1} space depending upon the genus of the elastica. Linear independence of these bases are guaranteed by the above truncation.

Then, for a curve $C \in \mathfrak{M}^{(g)}_{\mathrm{qcl}}$, the heat fluctuation of higher order is expressed as

$$\delta^{(n)} E[C, \delta t_{2m+1}] = \sum_{0<m_i\leqslant g} \frac{1}{\sqrt{\beta}^n} \prod_i^n (\delta t_{2m_i+1}) \int \mathrm{d}s \prod_i^n (\partial_{t_{2m_i+1}}) k^2. \tag{3.48}$$

Here the $m_i = 0$ part vanishes due to the periodicity

$$\int \mathrm{d}s\, \partial_s \left(\prod_i^{n-1} (\partial_{t2m_i+1} k^2) \right) = 0. \tag{3.49}$$

On the other hand, if the set $\{m_i\}$ does not contain $m_i = 0$ as a component, the integral commutes with these derivatives

$$\int \mathrm{d}s \prod_i^n (\partial_{t_{2m_i+1}}) k^2 = \prod_i^n (\partial_{t_{2m_i+1}}) \int \mathrm{d}s\, k^2. \tag{3.50}$$

Since $\int k^2\,\mathrm{d}s$ is invariant for the time $t_{2n+1}(n > 0)$ development from the soliton theory, (3.50) vanishes. Hence, we obtain that all higher-order fluctuations vanish

$$\delta^{(n)}E[C, \delta t_{2m+1}] \equiv 0 \qquad \text{for } n > 0. \tag{3.51}$$

In other words, the effect of heat fluctuation is given only through the energy functional of the quasi-classical motions and their volume. As mentioned before, the volume of a functional space is related to the entropy of the system. The functional space with the same energy is given as trajectory spaces of $\boldsymbol{t}_m$. Since the problem of the MKdV equation is an initial value problem in $\boldsymbol{t}_m$, we can choose any regular curve satisfying the boundary condition as an initial condition and then the set of trajectories of the solution of the MKdV equation exhibits the functional space, in which the elements have the same value as the energy. In other words, all conformations of the elastica can be classified by the MKdV equation and any conformation exists in the solution space of the MKdV equation, as least, as an initial condition. This means that the extension from one parameter t to infinite parameters and the choice of the minimal set (3.16) are justified and that they uniquely lead to the correct result. By investigation of its moduli, we sum the weight $\mathrm{e}^{-\beta E}$ over all conformations or the moduli of the elastica.

Since for a quasi-motion of genus g, the curvature determined from (3.3) is precisely given as

$$k(s, t_1, t_2, \ldots, t_{2g-1}) = k_{\mathrm{qcl}}\left(s, t_1 + \frac{1}{\sqrt{\beta}}\delta t_1, t_2 + \frac{1}{\sqrt{\beta}}\delta t_2, \ldots, t_{2g-1} + \frac{1}{\sqrt{\beta}}\delta t_{2g-1}\right) \tag{3.52}$$

we must integrate the Boltzmann weight $\exp(-\beta\int \mathrm{d}s\, k^2)$ over all δt except δt_1. Using the translation symmetry and freedom of the integration variable, we rewrite $t_{2n+1} = \delta t_{2n+1}/\sqrt{\beta}$ and replace the general curvature k with k_{qcl}.

Consequently, we obtain an explicit form of the regularized partition function (3.26), which is expressed by

$$\begin{aligned} Z_{\mathrm{reg}}[\beta] &= \sum_g \sum_{C\in\mathfrak{M}_{\mathrm{qcl}}^{(g)}} (\exp(-\beta E_{\mathrm{qcl}}[C])) \\ &= \sum_{g=0}^{1}\sum_{n\in\mathfrak{X}_g^0}(\exp(-\beta E_{\mathrm{qcl}}[C^{(n)}])) + \sum_{g=2}^{\infty}\sum_{n\in\mathfrak{X}^{(g)}}\int_{\mathfrak{M}^{(g)}_{t_m=0,n}} \mathrm{d}\mathfrak{m}\int_{\tilde{\Sigma}_{\mathfrak{m},n}}\left(\prod_{n=2}^{g}\mathrm{d}t_{2n-1}\right) \\ &\quad\times \exp(-\beta E_{\mathrm{qcl}}[C_{\mathfrak{m}}^{(n)}(\boldsymbol{t}_{\mathfrak{m}}^{(n)})]). \end{aligned} \tag{3.53}$$

This is the exact form of the partition function (3.1) of the non-stretching elastica without divergence. In the second term, there appears integration of the type $\int \mathrm{d}x\, \mathrm{e}^{-\beta f(x)}$. Thus, it is expected that the prefactor of the second term begins with the negative power of β. For large β, the second term is less than the first term. Hence, for the zero-temperature limit $\beta \to \infty$, the second term disappears and only the contribution of the genus zero and one survives. Noting that the moduli of the quasi-classical elastica with $g \leqslant 1$ is equivalent to that of the classical, we obtain

$$\lim_{\beta\to\infty} Z_{\mathrm{reg}}[\beta] = \max_{C\in\mathfrak{S}_{\mathrm{cls}}} \exp(-\beta E_{\mathrm{cls}}[C]) = \exp\left(-\beta \min_{C\in\mathfrak{S}_{\mathrm{cls}}} E_{\mathrm{cls}}[C]\right). \tag{3.54}$$

Depending upon the boundary condition, the classical solutions appear as miminal points of the partition function $Z_{\mathrm{reg}}[\beta]$. Hence, the partition function (3.53) does not contradict the discovery of Daniel Bernoulli [1].

4. Discussion

It is worthwhile noting that due to the isometric condition, we have derived the MKdV hierarchy. In the elastic body, the Lagrangian coordinate system should be employed rather than the Eulerian coordinate system when we use the terms of fluid mechanics. In elastic body theory by marking some points on an elastic body and by estimating the variation of distance among the marking points, which is measured using the induced metric, the force will be locally evaluated as a linear response for its certain deformation. The marking points correspond to the Lagrangian test particle in the language of fluid mechanics. On the other hand, as we have used the metric induced from the base space and calculated the deformation, our calculation corresponds to the Eulerian one. Here it should be noted that if one uses the induced metric or Eulerian picture, any stretching (physical) curve can be regarded as a non-stretching (mathematical) curve; it is a trivial trick between the Lagrangian picture and Eulerian picture and such recognition has few physical meanings. If stretching has a physical meaning like an elastic body, the Eulerian picture does not exhibit the dynamical situation and unless stretch plays an important role like the boundary curve of a binary fluid, dealing with stretch has less physical meaning. Accordingly, the isometric condition employed plays the central role in this scheme. In other words, in the above computation, the reason why we could physically use the Eulerian picture even in the elastic body problem is due to this isometry condition.

It should be also noted that even though there appears a nonlinear differential equation in this scheme, we have used the energy functional which is locally given in the framework of the linear response of the force for the deformation [4, 11]; if one uses the nonlinear energy functional, we must evaluate it from basic elastic body theory because it is beyond the ordinary elastic body theory. It is remarked that due to the bilinearity of the energy functional, which is established in the framework of the ordinary elastic body theory, we were able to find the exact partition function (3.53) in this model.

Furthermore, the origin of the MKdV hierarchy in [8, 9, 33] was artificial and was not physically supported. If one physically sets up a problem of time development of the elastica for real physical time, we conclude that its motion is not governed by the MKdV equation nor the MKdV hierarchy in general [10–14]. However, in this paper, we have obtained the MKdV hierarchy from the physical requirement and a (mathematical) parameter time δt_{2n+1} appears of variational direction as pointed out in [10]. In other words, by virtue of the novel investigation of the properties of the isometric curve of Goldstein and Petrich [8, 9], we conclude that the virtual dynamics is realized as a thermal fluctuation of an elastica in a heat bath. Due to the isometry condition, these equations become nonlinear differential equations. In the linear differential equation such as the harmonic oscillator, the mode, which is determined by the global feature of the system, is represented by a vector of momentum space. As well as mode analysis of the linear system, these parameters t_{2n+1} exhibit the global deformation of the elastica due to thermal fluctuation and they are expressed as vectors of the Jacobi theory.

It is remarked that the obtained partition function (3.53) differs from that in [24], which is obtained by summing the weight function over the conformation including non-isometry deformation. Due to the isometry condition, nonlinear terms appear in the quasi-classical curve equation while the partition function proposed by Saitô *et al* [24] is essentially linear. However, for perturbative deformation, for example from the circle, the nonlinear term might be negligible. Thus, as long as the deformation is perturbative, their partition function can be applicable for a polymer which cannot stretch even in thermal fluctuation.

On the other hand, our partition function is justifiable even for large deformation. The partition function is summed over different topology g, which is related to the writhing number of the conformation. Hence, there is a possibility of a topology change due to thermal fluctuation. It is of interest to calculate the possibility (or kernel function) from $g = 1$ conformation to $g = 2$ conformation. Even though the partition function (3.53) has not been concretely calculated, such computation, in principle, can be performed.

Next, we comment on the physical meaning of δt_{2n+1} and the relation between the BRS transformation [31] and the Sato coordinate [40–42]. Since we have dealt with the SO(2) principal bundle over S^1, the gauge group is expressed as

$$\mathfrak{G} \subset \coprod_{s\in S^1} \mathrm{SO}(2) \tag{4.1}$$

where $\coprod$ means the disjoint union. $\mathfrak{G}$ is the infinite-dimensional Lie group. It acts upon the shape of the curve, which corresponds to a section of the principal bundle, and deforms it. For a given shape of elastica with a genus g, there is a unique group element which acts upon the elastica to become the shape with constant curvature, i.e. the simplest classical solution with $g = 0$. Thus the genus is well defined, which is induced from the genus of curve (the quasi-classical section). There is a filtered decomposition of $\mathfrak{G}$ as a family of subgroups $\mathfrak{G}_g$ with respect to the genus, whose action on the elastica preserves its genus. The representation of each group $\mathfrak{G}_g$ will be realized as the $\mathfrak{G}_g$ module in the set of corresponding Jacobi varieties. However, in soliton theory, instead of dealing with individual sets of Jacobi varieties of genus g, it is natural to consider the UGM if one wishes to formally treat a soliton equation. In fact, there are singular elements in $\mathfrak{G}$, which change the genus of the elastica; such transformations are known as global gauge transformations. Corresponding to UGM, $\mathfrak{G}$ should also be regarded as the inductive limit of the filtration of $\mathfrak{G}_g$ with respect to the genus g and then $\mathfrak{G}$ naturally contains the singular elements due to the natural extension of the group action. Thus $\mathfrak{G}$ is represented as a subset of $\mathrm{GL}(\infty)$ in the UGM. The quasi-classical curve (a section of the SO(2) principal bundle) is embedded in the UGM. The infinitesimal deformation of the curve in the UGM can be expressed by the vector in the UGM. In other words, such deformation exhibits (mathematical) velocity of a trajectory in the UGM and can be represented as a subset of the infinite-dimensional general linear Lie algebra $\mathrm{gl}(\infty)$, which is known as the affine Lie algebra $A_1^{(1)}$ [41]; $A_1^{(1)}$ is the Lie algebra associated with the Lie group $\mathfrak{G}$. We introduce the extrinsic differential operator in the UGM

$$\delta := \sum_n \mathrm{d}t_{2n+1}\partial_{t_{2n+1}} \qquad \delta^2 = 0. \tag{4.2}$$

Then (3.15) and (3.17) are expressed as for $A := k_{\mathrm{qcl}}\,\mathrm{d}s$

$$\delta A = \tilde{\Omega}\mathfrak{u}_1 \tag{4.3}$$

where

$$\mathfrak{u}_1 = \sum_n u_1^{(n)}\,\mathrm{d}t_{2n+1} \qquad \tilde{\Omega} := \Omega k_{\mathrm{qcl}}^{-1}\partial_s. \tag{4.4}$$

Noting the fact that $\mathfrak{u}_1^{(n)}$ is the Hamiltonian density of the MKdV hierarchy

$$\delta\mathfrak{u}_1 \approx 0 \tag{4.5}$$

where $\approx$ means equivalence after integration of both sides over s like (3.49) and (3.50). Since $\mathfrak{u}_1$ obeys the Grasmannian algebra, $\mathfrak{u}_1$ can be regarded as a fermionic field over S^1. Consequently (4.3) and (4.5) can be regarded as the BRS transformation of this system.

Hence, δt_{2n+1} in the path integration may be naturally understood in the framework of the Faddeev–Popov integration scheme [31]. In fact, the square root of the Frenet–Serret system (2.2) can be regarded as the Dirac operator [11, 12], which is realized by confining the free Dirac field into a thin elastic rod. Using the Dirac field confined in the elastica, we have constructed the MKdV hierarchy and τ-function as the partition function of the Dirac field [47]. Thus, it is expected that the partition function (3.53) should also be expressed by the τ-function of the MKdV hierarchy.

Here, we mention a conjecture associated with the critical phenomenon of this elastica model. The critical point must be determined as a topological discontinuity of the moduli space of the quasi-classical elastica. At this point, physical quantity sometimes diverges and becomes meaningless. We expect that the length of the elastica becomes less important around the critical point, for example the topological change, as a kernel function at an ordinary second-order critical point becomes scale-invariant [48]. We will consider a dilatation of a quasi-classical elastica for the normal direction of elastica

$$X_{\mathrm{c}} = X + \mathrm{i}t\, \mathrm{e}^{\mathrm{i}\phi_{\mathrm{c}}} \tag{4.6}$$

where X_{c} and ϕ_{c} are an affine vector and tangential phase of the quasi-classical elastica at the critical point, and t is a real deformation parameter. The infinitesimal length of the elastica $\mathrm{d}s_{\mathrm{c}}$ becomes

$$\mathrm{d}s_{\mathrm{c}} = \sqrt{\mathrm{d}X_{\mathrm{c}}\, \mathrm{d}\bar{X}_{\mathrm{c}}} \approx (1 - k_{\mathrm{c}}t)\, \mathrm{d}s \qquad k_{\mathrm{c}} := \partial s \phi_{\mathrm{c}} \tag{4.7}$$

and then the length of elastica is

$$\int \mathrm{d}s_{\mathrm{c}} \approx L - t(\phi_{\mathrm{c}}(L) - \phi_{\mathrm{c}}(0)). \tag{4.8}$$

Noting that $\phi_{\mathrm{c}}(L) - \phi_{\mathrm{c}}(0) = 2\pi n$, $n \in \mathbb{Z}$, this deformation makes the length of the elastica change. However, this deformation must be compatible with the isometric condition since we have been dealing with non-stretching elastica. Both requirements seem to contradict each other, but the critical point is an irregular point at which the contradicted objects coexist. From (3.2), it in (4.6) must be proportional to $u_{\mathrm{c}} := u_1 + \mathrm{i}u_2$ with (3.18b), and u_1 satisfies (3.19b) in respect of the deformation parameter t

$$\mathrm{i}ct = u_{\mathrm{c}} = \tfrac{1}{4}k_{\mathrm{c}}^2 - \mathrm{i}\tfrac{1}{2}\partial_s k_{\mathrm{c}} \tag{4.9}$$

where c is a real proportional constant. Since this relation is the Miura map, from (3.19b), u_{c} obeys the KdV equation

$$\mathrm{i}c = \partial_t u_{\mathrm{c}} = 6u_{\mathrm{c}}\partial_s u_{\mathrm{c}} + \partial_c^3 u_{\mathrm{c}}. \tag{4.10}$$

By integrating it in s, (4.10) becomes

$$\mathrm{i}cs = 3u_{\mathrm{c}}^2 + \lambda\partial_s^2 u_{\mathrm{c}}. \tag{4.11}$$

For $z := \mathrm{i}sw := u_{\mathrm{c}}/2$ and $c = -2$, (4.11) can be rewritten as

$$\partial_z^2 w = 6w^2 + z. \tag{4.12}$$

This is the Painlevé equation of the first kind. Thus, we conjecture that at a critical point of the elastica, an expectation value obeys the Painlevé equation of the first kind [49]. In our scheme, we formally obtain the series of the ordinary differential equations related to the KdV hierarchy (from the Miura map) and the Painlevé transcendents. Thus, the moduli space must be closely related to the quantum two-dimensional gravity, in which (4.12) and the KdV hierarchy naturally appear [50–52]. In fact, the loop soliton partially appears in the immersed surface in three-dimensional space $\mathbb{R}^3$ [38, 53]. By the fermionic study

[47, 53–57], the immersed surface system is interpreted as a natural extension of the elastica system and also that of the Liouville surface system whose quantum version is known as the quantum two-dimensional gravity [55, 56]. Thus, we plan to investigate the immersed surface system and reveal the relation between the elastica system and the quantum two-dimensional gravity. (Here it should be noted that the elastica problem is not directly related to the string problem in string theory because the action of the elastica is biharmonic for X while that of the string is harmonic [55]. In non-relativistic space, thickness is more important and the biharmonic equation is very natural.)

As we dealt with the kinetic properties of the large closed polymer and investigated the moduli of the MKdV equation in $\mathbb{C}P^1$, recently another statistical mechanics model of a large polymer has been reported [58]. A partition function of non-contractible self-avoiding two-dimensional polymers in the topological torus

$$T = \mathbb{C}/(L_1\mathbb{Z} + L_2\mathbb{Z}) \tag{4.13}$$

was studied associated with the MKdV equation. The partition function of such polymers was also solved by the MKdV equation [58] and is recognized as its τ-function [59]. As are assumed the base space as $\mathbb{C}P^1$, it can also be replaced with the topological torus. Then, the ratio $L : L_1 : L_2$ becomes important as the boundary conditions but our partition function can be formally calculated. The obtained partition function may also be closely related to the τ-function of the MKdV equation. It is a very interesting fact that the MKdV equation appears and plays central roles in both theories even though both are not directly concerned with the models.

Finally, we will mentioned further possibilities and the development of this theory. We have investigated the elastica in two-dimensional space but can extend this theory to that in three-dimensional space if we can classify the solutions of quasi-classical curves of the elastica in three-dimensional space. It is well known that the nonlinear Schrödinger equation appears as a vortex soliton in three-dimensional space, which is related to an elastica without the elastic torsion force [60]. Using the similarity between the MKdV equation and the nonlinear Schrödinger equation, the study can be performed by imitating the above argument [15, 28, 63]. Then, instead of the Goldstein–Petrich scheme [8, 9], there appears the Langer–Perline scheme [62] which is the higher-dimensional object of the Goldstein–Petrich one [15]. Then the nonlinear Schrödinger hierarchy appears as a substitute of the MKdV hierarchy [15, 61–63]. There we might also need investigation into the moduli of quasi-classical elastica in three-dimensional space including topological properties like a knot invariance. Then the ambient isotopy plays an important role besides the fundamental group (3.23) [64].

Furthermore, even though we have formally classified the moduli of the quasi-classical elastica in two-dimensional space, we cannot explicitly draw the shape of the closed elastica of $g > 1$ now. Accordingly, it is very important for this study to find explicit shapes of closed elastica of $g > 1$. If we could explicitly draw any shapes of closed quasi-classical elastica in the plane of $g = 2$ or 3 using hyperelliptic functions, we think they may have enormous effects on this problem. Since the summation of the partition function might converge on the genus g, to determine the shapes even for small g means important steps towards solving this problem.

Moreover, it is also of interest to deal with an elastic rod which can stretch as a more physical model [14] and higher-dimensional objects, such as an immersed surface [38, 53, 56, 57, 65].

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